For all questions, answer choice E) NOTA means that none of the above answers is correct.

1. **C)** Since $f(x,y)=x^2+y^2, y(0)=-1$ and $h=\frac{1}{2}$. Use Euler's method, we have

$$y(h)=y(0)+hf(0,y(0))=-1+\frac{0^2+(-1)^2}{2}=-\frac{1}{2}$$

$$y(2h)=y(h)+hf(h,y(h))=-\frac{1}{2}+\frac{\frac{1}{2}^2+(-\frac{1}{2})^2}{2}=-\frac{1}{4}$$

$$y(3h)=y(2h)+hf(2h,y(2h))=-\frac{1}{4}+\frac{1^2+(-\frac{1}{4})^2}{2}=\frac{9}{32}$$

The correct answer is C.

2. **C)** Since $f(x,y)=y^2+2x, y(0)=-1$ and $h=\frac{1}{2}$. Use Euler's method, The end of this Euler strut is $(h,y(h))=(h,y(0)+hf(0,y(0)))=(\frac{1}{2},-\frac{1}{2})$ The first slope is the average of

$$\frac{f(0,y(0))+f(h,y(h))}{2}=\frac{f(0,-1)+f(\frac{1}{2},-\frac{1}{2})}{2}=\frac{1+\frac{5}{4}}{2}=\frac{9}{8}.$$

The correct answer is C.

3. **C)** It is easy to see $y'=\frac{x^2}{1-3y^2}\Rightarrow(1-3y^2)y'=x^2\Rightarrow 3(y-y^3)=x^3+c$. Condition $y(\sqrt[3]{3})=-2$ implies $c=15$ so the solution is $3(y-y^3)=x^3+15$. $y(-\sqrt[3]{15})=0=3y(1-y^2)\Rightarrow y=0,-1,1$.
The correct answer is C.

4. **D)** Let $x=t^2, y=u^2$, then $y'=\frac{\sin\sqrt{x}}{\sqrt{y}}$ can be rewritten as $\frac{uu'}{t}=\frac{\sin t}{u}$. The general solution is $\frac{u^3}{3}=\sin t-t\cos t+c$ which means $\frac{y^{\frac{3}{2}}}{3}=\sin\sqrt{x}-\sqrt{x}\cos\sqrt{x}+c$. Condition $y(0)=3^{\frac{2}{3}}$ implies $c=1$, so the solution is $\frac{y^{\frac{3}{2}}}{3}=\sin\sqrt{x}-\sqrt{x}\cos\sqrt{x}+1$. $y(\frac{\pi^2}{4})=6^{\frac{2}{3}}$. The correct answer is D.

5. **E)**

6. **A)** The integrating factor of given equation is $e^{\cos x}$. Multiply $e^{\cos x}$ and integrating the given equation, we have the general solution is $ye^{\cos x}=c-2e^{\cos x}$. Condition $y(\frac{\pi}{2})=1$ implies $c=3$ so the solution is $(y+2)e^{\cos x}=3$. It is easy to see $(y+2)e^{\cos x}=3$ passes through point $(-\frac{\pi}{2},1)$. The correct answer is A.

7. **D)** The integrating factor is $e^{\int -2dx}=e^{-2x}$. The correct answer is D

8. **A)** The integrating factor is $e^{\int \frac{1}{y}dy}=y$. The general solution is $xy=c+y^2$. Condition $(0,-1)$ implies $c=-1$ so the solution is $xy=1-y^2$. It is easy to see $x(1)=0$. The correct answer is A

9. **C)** The integrating factor is $e^{\int \frac{2}{x}dx}=x^2$. The general solution is $xy^2=10\int_0^x \frac{\sin t}{t}dt+c$. Using condition $(1,0)$ implies $c=-10\int_0^1 \frac{\sin t}{t}dt=-10\mathrm{Si}(1)$ so the solution is $xy^2=10(\mathrm{Si}(10)-\mathrm{Si}(1))$. The correct answer is C.

10. **C)** The general solution is $2\sin xy=(x+1)^2+c$. Using condition $(0,\frac{\pi}{2})$ implies $c=-1$ so the solution is $2\sin xy=(x+1)^2-1$. It is easy to see $y(0)=1, y(-1)=\frac{\pi}{6}$. The correct answer is C.

11. **A)** Since $xy'+y=xy^3 \Rightarrow \frac{y'}{y^3}+\frac{1}{xy^2}=1 \Rightarrow -2\frac{y'}{y^3x^2}-2\frac{1}{y^2x^3}=\frac{-2}{x^2} \Rightarrow (\frac{1}{y^2x^2})'=\frac{-2}{x^2} \Rightarrow \frac{1}{y^2x^2}=\frac{2}{x}+c$
Using condition $(1,1)$ implies $c=-1$ so the solution is $y^2x(2-x)=1$. The correct answer is A.

12. **D)** Rewrite the given equation in the standard form $y'=\dfrac{ay-3x+\frac{y^2}{x}}{x-y}$. So the function f in this case is given by $f(x,y)=\dfrac{ay-3x+\frac{y^2}{x}}{x-y}$. The given equation is Euler homogeneous if and only if $f(x,y)$ is scale invariant. It is easy to see $f(cx,cy)=\dfrac{c(ay-3x+\frac{y^2}{x})}{c(x-y)}=f(x,y)$ for any real number a. The correct answer is D.

13. **B)** Rewrite the given equation in the form $\frac{(y+1)'}{(y+1)^2}-\frac{1}{x(y+1)}=\frac{1}{x^2}$. The general solution is $-\frac{x}{(y+1)}=\ln|x|+c$. Condition $(-1,0)$ implies $c=1$ so $-\frac{x}{(y+1)}=\ln|x|+1$, $y(1)=-2$.
The correct answer is B.

14. **B)** It is easy to see $N(x,y)=x^2+2y, M(x,y)=axy+b\sin x$.The given equation is an exact if and only if $\frac{\partial M}{\partial y}=ax=\frac{\partial N}{\partial x}=2x$. Therefore, constant a = 2. B is correct answer.

15. **B)** It is easy to see $N(x,y)=x^2e^y+\cos x-1, M(x,y)=x^2e^y-y\sin x$.The equation is exact since $\frac{\partial M}{\partial y}=2xe^y-\sin x=\frac{\partial N}{\partial x}$. $x^2e^y+y\cos x-y=c$ is the general. B is correct answer.

16. **B)** It is easy to see $N(x,y)=x^2+xy, M(x,y)=3xy+y^2$ and $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{3x+2y-2x-y}{x(x+y)}=\frac{1}{x}$ which is y independent. The integrating factor is x. B is correct answer.

17. **C)** Since $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=\frac{4x-x}{xy}=\frac{3}{y}$ which is x independent. The integrating factor can be y^3. After we multiply the given equation by y^3, the resulting equation is $xy^4dx+(2x^2y^3+3y^5-20y^3)dy=0$. The general solution is $x^2y^4+y^6-10y^4=c$. C is correct answer.

18. **D)** $x^2y''-3xy'+4y=0$ can be rewritten in standard form $y''-\frac{3}{x^2}y'+\frac{4}{x^2}y=0$. Use $y_2=y_1\int\frac{e^{\int -pdx}}{y_1^2}dx$, we have $y_2=y_1\int\frac{e^{\int -pdx}}{y_1^2}dx=x^2\int\frac{e^{\int\frac{3}{x}dx}}{x^4}dx=x^2\ln x$. D is correct answer.

19. **E)** The auxiliary equation of given equation is $r''+4r-5=0$ has roots $r=1 \quad or\ r=-5$. The general solution is $y=Ae^x+Be^{-5x}$. Initial conditions implies $A=\frac{11}{3}, B=\frac{1}{3}$, we have $y=\frac{11}{3}e^x+\frac{1}{3}e^{-5x}$. No correct answer was provided so the correct answer is E.

20. **D)** The auxiliary equation of given equation is $r''+4=0$ has roots $r=2i \quad or\ r=-2i$. The general solution is $y=A\cos 2x+B\sin 2x$. Initial condition only implies $B=2$, we have $y=A\cos 2x+2\sin 2x \quad y'(0)=4$. D is correct answer.

21. **C)** It is easy to see $y=\pm 1$satisfies equation and $y'^2+y^2-1=0 \Rightarrow y'=\pm\sqrt{1-y^2}$. Integrating $y'=\sqrt{1-y^2}$, we have $\sin^{-1}y=x+c$. Integrating $y'=-\sqrt{1-y^2}$, we have $\cos^{-1}y=x+c$. D is correct answer.

22. **C)** The critical point is $f(x,y)=(y+1)(y-2)(4-y)=0 \Rightarrow y=-1, y=2, y=4$. C is correct answer.

23. **D)** From $y''=x+y-y^2, y(0)=-1, y'(0)=1$, we have $y''(0)=-2$ and $y'''=1+y'-2yy' \Rightarrow y'''(0)=1+1-2(-1)1=4$. D is correct answer.

24. **A)** From $y'y''=x, y(0)=1, y'(1)=\sqrt{2}$, we have $y'^2=x^2+c$ by integrating. Substituting $y'(1)=\sqrt{2}$ into general solution implies c = 1. That is $y'^2=x^2+1$. On the other hand, we have $y''=\frac{x}{y'} \Rightarrow y'''=\frac{y'-xy''}{y'^2}$. Therefore, $y'''(0)=\frac{1}{y'(0)}=1$. A is correct answer.

25. **D)** From $y''=1-y'^2$, we have $y'''=-2y'y''=2y'(y'^2-1) \Rightarrow y^{(4)}=-2(1-y'^2)^2+4y'^2y''=2(1-y'^2)(3y'^2-1)$ Therefore, $y^{(4)}(0)=2(1-2)(3\cdot 2-1)=-10$. D is correct answer.

26. **B)** From given equation, we have $y'=1+2x+3x^2+4x^3+\cdots$.

$$
\begin{aligned}
y^2 = 1&+x+x^2+x^3+\cdots \\
&+x+x^2+x^3+\cdots \\
&\quad +x^2+x^3+\cdots \\
&\qquad +x^3+\cdots \\
&\vdots
\end{aligned}
$$

Therefore, $y'=y^2$, $y''=2yy'=2y^3, y'''=6y^2y'=6y^4, \cdots y^{(n)}=n!\,y^{n+1}$. B is correct answer.

27. **B)** The auxiliary equation of given equation is $r^2-2r+1=0$ has roots $r_1=1=r_2$. We assume the form of particular solution is $y_p=Ax^2x^x$. Substituting into given equation, we have $y_p=\frac{x^2}{2}e^x$. B is correct answer.

28. **E)** From the given equation $y=(x+1)y'+y'^2$, we have $0=(x+1)y''+2y'y''$, which implies $y'=c$ and $x+1+2y'=0$. The solution is $y=(x+1)c+c^2$ or $y=-\frac{(x+1)^2}{4}+A$. Condition $y(1)=-1$ implies $A=0, c=-1$, The solution is $y=-x$ or $y=-\frac{(x+1)^2}{4}$. $y=-x$ passes points $(0,0)$ and $(2,-2)$ while $y=-\frac{(x+1)^2}{4}$ passes points $(-1,0)$. E is correct answer.

29. **A)** The auxiliary equation is $r''' + r'' = 0$ has roots $r_1 = r_2 = 0, r_3 = -1$. We assume the given equation has a particular solution in the form $y_p = (A\sin x + B\cos x)e^x$. Substituting into the given equation, we have $y_p = (\frac{2}{5}\sin x + \frac{3}{10}\cos x)e^x$. A is correct answer.

30. **D)** From given equation $y' + y\cot x = 2\cos x, y(\frac{\pi}{2}) = 5$, we have $y\sin x = \frac{c - \cos 2x}{2}$. Condition $y(\frac{\pi}{2}) = 5$ implies $y\sin x = \frac{9 - \cos 2x}{2}$, $y(\frac{\pi}{2}) = 3$ implies $y_T \sin x = \frac{5 - \cos 2x}{2}$.
$(y(x) - y_T(x))|_{x=\frac{\pi}{6}} = 4$. The correct answer is D.